

Determinant

Determinant of 1×1 matrix, $\mathbf{A} = [a_{11}]$

$$|a_{11}| = a_{11}$$

Determinant of 2×2 matrix, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of any order n

$$\text{Determinant, } D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

For $n \geq 2$,

$$D = \sum_{k=1}^n (-1)^{j+k} a_{jk} M_{jk} \quad ; (j=1,2,\dots, \text{orn})$$

or

$$D = \sum_{j=1}^n (-1)^{j+k} a_{jk} M_{jk} \quad ; (k=1, 2, \dots, \text{orn}) .$$

Where M_{jk} (is called minor of a_{jk}) is a determinant of order $n-1$, that is the determinant of the sub matrix of \mathbf{A} obtained from \mathbf{A} by deleting the row and column of the entry a_{jk} (the j^{th} row and the k^{th} column).

For instance, $n=3$,

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{k=1}^3 (-1)^{j+k} a_{jk} M_{jk} \quad ; (j=1,2,\text{or}3)$$

choose $j=1$,

$$\begin{aligned} D &= \sum_{k=1}^3 (-1)^{1+k} a_{jk} M_{jk} = (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + (-1)^{1+3} a_{13} M_{13} \\ &= a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

Remarks

This is called expand the determinant through the 1st row. You may expand the determinant through any row or column.

Remarks

1. the cofactors of a_{jk} in \mathbf{A} is defined to be $A_{jk} = (-1)^{j+k} M_{jk}$,
2. so, we can write the determinant of an $n \times n$ (for $n \geq 2$) matrix \mathbf{A} as:

$$|\mathbf{A}| = \sum_{k=1}^n a_{jk} A_{jk} \quad ; (j=1, 2, \dots, \text{orn}) \text{ or}$$

$$|\mathbf{A}| = \sum_{j=1}^n a_{jk} A_{jk} \quad ; (k=1, 2, \dots, \text{orn}) .$$

Determinant of a Triangular Matrix

The determinant of any triangular matrix equals the product of all the entries on the main diagonal. To see this, expand by row if the matrix is lower triangular matrix, and by column if it is upper triangular matrix.

For instance,

$$\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix} = -3 \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} = -3 \cdot 4 \cdot 5 = -60 .$$

General Properties of Determinants

Since the same value is obtained whether we expand a determinant by any row or any column, we have,

Theorem 1 (Transposition)

The value of a determinant is not altered if its rows are written as columns, in the same order.

Example:

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 3 \begin{vmatrix} 1 & 2 & -1 \\ 6 & 6 & 0 \\ 0 & 4 & 2 \end{vmatrix} = -12$$

Theorem 2 (Multiplication by a constant)

If all the entries on one row or one column of a determinant are multiplied by the same factor k , the value of the new determinant is k times the value of the given determinant.

Proof: Expand the determinant by that row (or column) whose entries are multiplied by k .

Application of theorem 2:

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 & 0 \\ 1 & 3 & 2 \\ -1 & 0 & 2 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{vmatrix} = 12 \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = -12$$

From Theorem 2, with $k = 0$, or directly by expanding, we obtain,

Theorem 3

If all the entries in a row (or a column) of a determinant are zero, the value of the determinant is zero.

Theorem 4

If each entry in a row (or a column) of a determinant is expressed as a binomial, the determinant can be written as the sum of two determinants.

Proof: Expand the determinant by the row (or column) whose entries are binomials.

Example: Illustration of theorem 4.

$$\begin{vmatrix} a_1 + d_1 & b_1 & c_1 \\ a_2 + d_2 & b_2 & c_2 \\ a_3 + d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Theorem 5 (Interchange of rows or columns)

If any two rows (or two columns) of a determinant are interchanged, the value of the determinant is multiplied by -1 .

Example: Interchange of two rows

$$\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 0 \\ -1 & 0 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 12$$

Theorem 6: (Proportional rows or columns)

If corresponding entries in two rows (or two columns) of a determinant are proportional, the value of the determinant is zero.

Proof: Let the entries in the i^{th} and j^{th} rows of D be proportional, say, $a_{ik} = ca_{jk}$, $k = 1, \dots, n$. If $c = 0$, then $D = 0$. Now let $c \neq 0$. By theorem 2,

$$D = cB$$

where the i^{th} and j^{th} rows of B are identical. Interchange these rows. Then B goes over into $-B$, by theorem 5. On the other hand, since the rows are identical, the new determinant is still B . Thus $B = -B$, so $B = 0$, and $D = 0$.

Example: Proportional rows

$$\begin{vmatrix} 3 & 6 & -4 \\ 1 & -1 & 3 \\ -6 & -12 & 8 \end{vmatrix} = 0$$

Theorem 7 (Addition of a row or column)

The value of a determinant is left unchanged if the entries in a row (or column) are altered by adding to them any constant multiple of the corresponding entries in any other row (or column, respectively).

Proof: Apply Theorem 4 to the determinant that results from the given addition. This yields a sum of two determinants: one is the original determinant and the other contains proportional rows. According to theorem 6, the second determinant is zero, and the proof is complete.

Example: Evaluation of a determinant by reduction to “triangular form”

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix} \quad \begin{array}{l} R_3 \rightarrow R_3 - 2R_1 \\ R_4 \rightarrow R_4 - 1.5R_1 \end{array}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{vmatrix} \quad \begin{array}{l} R_3 \rightarrow R_3 - 0.4R_2 \\ R_4 \rightarrow R_4 - 1.6R_2 \end{array}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & 0 & 47.25 \end{vmatrix} \quad R_4 \rightarrow R_4 + 4.75R_3$$

$$= 2 \times 5 \times 2.4 \times 47.25 = 1134$$

In work with pencil and paper, one write down lower order determinants when they appear, instead of carrying along zeros,

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix} = \dots 2 \begin{vmatrix} 5 & 9 & -12 \\ 2 & 6 & -1 \\ 8 & 3 & 10 \end{vmatrix} = \dots 10 \begin{vmatrix} 2.4 & 3.8 \\ -11.4 & 29.2 \end{vmatrix} = 1134$$

Theorem 8 (Determinant of a product of matrices)

For any $n \times n$ matrices \mathbf{A} and \mathbf{B} ,

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det(\mathbf{A}) \det(\mathbf{B})$$

Example: Illustration of Theorem 8

$$\begin{vmatrix} 2 & 4 & 3 \\ 6 & 10 & 14 \\ 4 & 7 & 9 \end{vmatrix} \begin{vmatrix} 4 & 0 & 5 \\ -2 & 1 & -1 \\ 3 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 9 & 4 & 18 \\ 46 & 10 & 76 \\ 29 & 7 & 49 \end{vmatrix}$$

Theorem 9 (Derivative of a determinant)

The derivative D' of a determinant D of order n whose entry are differentiable functions can be written as

$$D' = D_{(1)} + D_{(2)} + \cdots + D_{(n)}$$

where $D_{(j)}$ is obtained from D by differentiating the entries in the j^{th} row.

Theorem 10 (Rank in Terms Determinant)

An $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ has rank r if and only if \mathbf{A} has an $r \times r$ submatrix with nonzero determinant, whereas the determinant of every square submatrix with order $r + 1$ or more that \mathbf{A} has (or does not have!) is zero.

In particular, if \mathbf{A} is a square $n \times n$ matrix,

- if $|\mathbf{A}| \neq 0$, then $\text{rank}(\mathbf{A}) = n$
- if $|\mathbf{A}| = 0$, then $\text{rank}(\mathbf{A}) < n$

A few more formulas:

1. For any constant c , $\det(c\mathbf{A}) = c^n \det(\mathbf{A})$ where \mathbf{A} is a $n \times n$ matrix.
2. For a matrix \mathbf{A}^{-1} such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$.

Cramer's Rule

If $\mathbf{A}\vec{x} = \vec{b}$ is a system of n equations in n unknowns, such that $|\mathbf{A}| \neq 0$ (this means $\text{Rank } \mathbf{A} = \text{Rank } \mathbf{A}_b$), then the system has a unique solution. The solution is $x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}$, ..., $x_n = \frac{D_n}{D}$, where D is equal to $|\mathbf{A}|$ and D_i is the determinant of the matrix obtained by replacing the i^{th} column of \mathbf{A} by the column of vector \vec{b} .

Remark

- From the Cramer's Rule, we can observe that if the system is homogeneous and $D \neq 0$, then it has only the trivial solution $x_1 = 0, x_2 = 0, \dots, x_n = 0$.
- If $D = 0$, the homogeneous system also has nontrivial solutions.

Determinants. Cramer's Rule

1. Evaluate the following determinants:

$$(a) \begin{vmatrix} \cos nq & \sin nq \\ -\sin nq & \cos nq \end{vmatrix}$$

$$(b) \begin{vmatrix} 1 & 205 & 479 \\ 0 & 2 & 103 \\ 0 & 0 & 3 \end{vmatrix}$$

$$(c) \begin{vmatrix} 7 & 14 & 21 \\ 36 & 18 & 6 \\ 87 & 12 & -45 \end{vmatrix}$$

$$(d) \begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 16 \end{vmatrix}$$

$$(e) \begin{vmatrix} 1 & 3 & 5 & 2 & 6 \\ -1 & 1 & 8 & 2 & 1 \\ 2 & -2 & -16 & -4 & 2 \\ 1 & 3 & -1 & 4 & 4 \\ 0 & -4 & -13 & -1 & 1 \end{vmatrix}$$

Hint:

Make use of the properties of determinant under elementary row operation to reduce the matrix to triangular form.

2. For $\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, assume that $|\mathbf{A}| = 10$.

Use the properties of determinant under the elementary row operation, find the value of the following determinant:

$$(i) \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + 2c_1 & b_2 + 2c_2 & b_3 + 2c_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(iii) \begin{vmatrix} -2a_1 & -2a_2 & -2a_3 \\ -2b_1 & -2b_2 & -2b_3 \\ -2c_1 & -2c_2 & -2c_3 \end{vmatrix}$$

3. Find the rank by determinants methods. Check your answer by row reduction methods.

(a)
$$\begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 21 & -3 & 17 & 13 \\ 46 & 11 & 52 & 14 \\ 33 & 48 & 71 & -23 \end{bmatrix}$$

4. Solve by Cramer's rule and check by Gauss elimination:

(a)
$$\begin{aligned} x + 2y + 3z &= 20 \\ 7x + 3y + z &= 13 \\ x + 6y + 2z &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 3x + 7y + 8z &= -13 \\ 2x + 9z &= -5 \\ -4x + y - 26z &= 2 \end{aligned}$$

Answer:

Question 1

(a) 1, (b) 6, (c) 0, (d) -64, (e) 192.

Question 2

(i) 10, (ii) 10, (iii) -80.

Question 3

(a) 1, (b) 3, (c) 2

Question 4

(a) $x = 2, y = -3, z = 8$

(b) $x = -7, y = 0, z = 1$